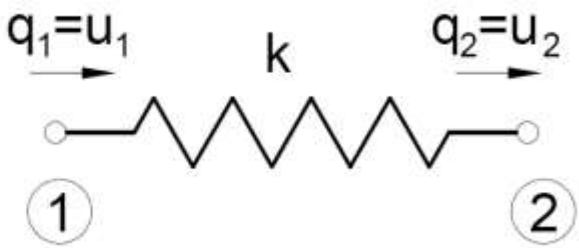
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 7A. Rod element

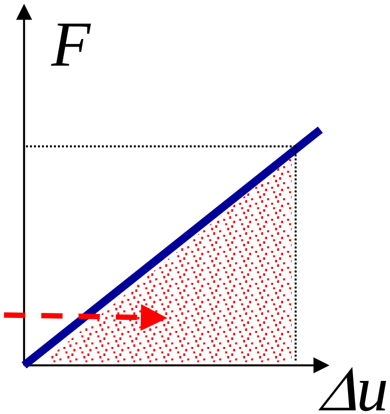
04.2025

Reminder - spring type element



vector of nodal parameters:

$$\underset{2\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



$$F = k \cdot \Delta u = k \cdot (u_2 - u_1)$$

elastic strain energy of the element:

$$U_e = \frac{1}{2} F \Delta u = \frac{1}{2} k (\Delta u)^2 = \frac{1}{2} k (u_2 - u_1)(u_2 - u_1).$$

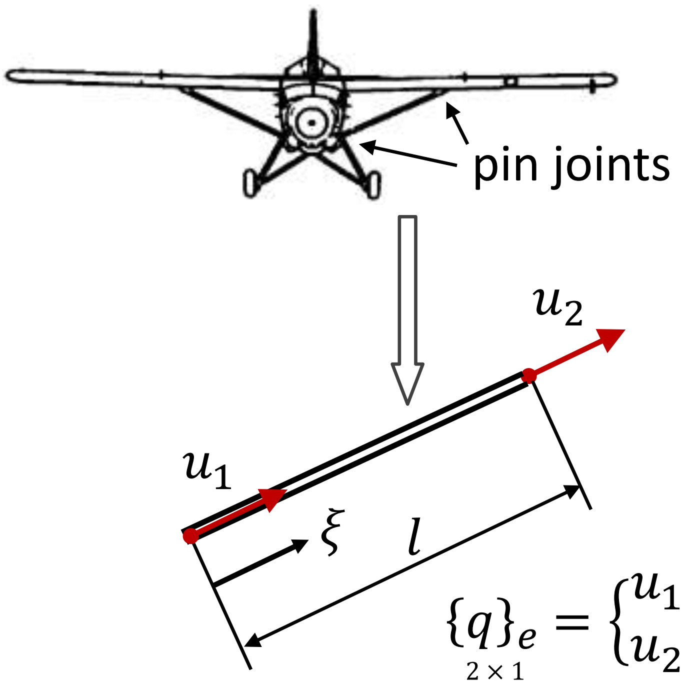
$$U_e = \frac{1}{2} \lfloor u_1, u_2 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e$$

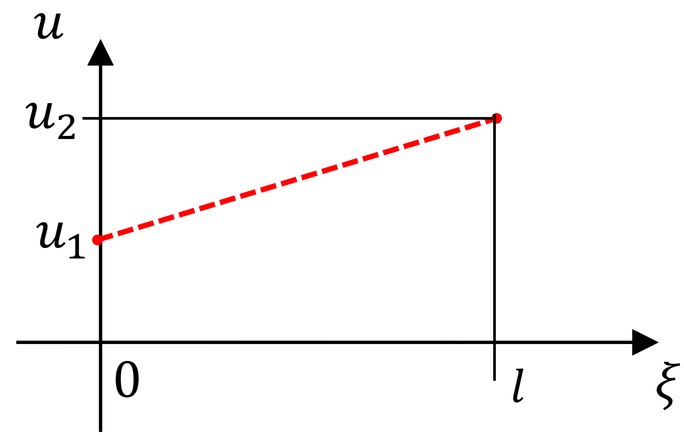
element stiffness matrix:

$$[k]_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Reminder - shape functions for a finite element representing a brace



$$\underset{2\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



linear functions:

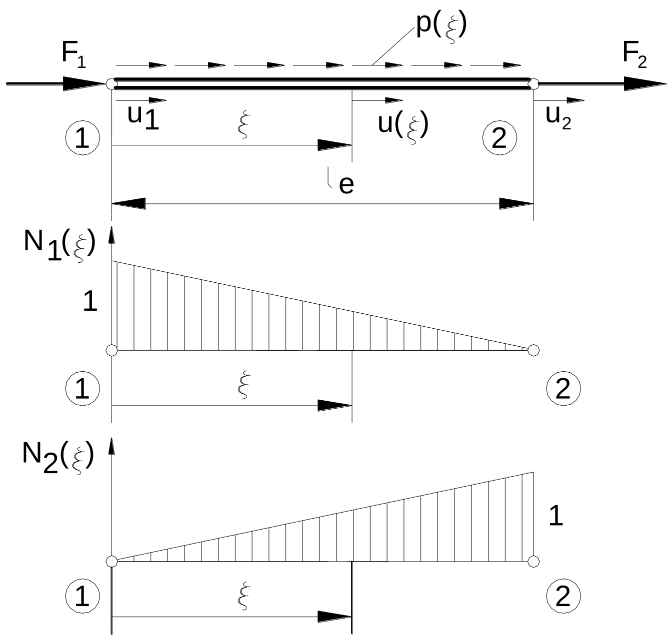
$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1$$

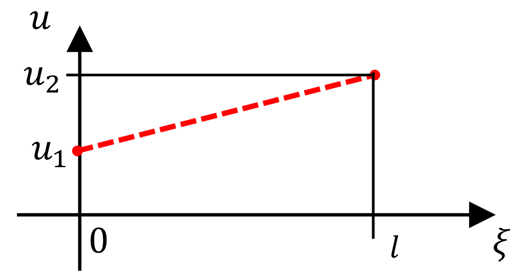
$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1 = \frac{u_2}{l}\xi - \frac{u_1}{l}\xi + u_1 = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 =$$

$$= N_1(\xi)\cdot u_1 + N_2(\xi)\cdot u_2 = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e = \underset{1\times 2}{\lfloor N(\xi) \rfloor} \underset{2\times 1}{\{q\}_e}$$

shape functions: $N_1(\xi) = 1 - \frac{\xi}{l}$; $N_2(\xi) = \frac{\xi}{l}$

Finite element of a rod under axial distributed loads





Axial displacement:

$$u(\xi) = u_1 + \frac{u_2 - u_1}{l_e}\xi$$

Vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$

Approximation of the axial displacements:

$$u(\xi) = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \lfloor N \rfloor \{q\}_e$$

Shape function vector:

$$\lfloor N \rfloor = \lfloor N_1(\xi), N_2(\xi) \rfloor$$

Shape functions:

$$N_1(\xi) = 1 - \frac{\xi}{l_e}, \qquad N_2(\xi) = \frac{\xi}{l_e}$$

Elastic strain energy of an element:

$$U_e = \frac{1}{2}A\int_0^{l_e} \sigma(\xi)\varepsilon(\xi)d\xi = \frac{EA}{2}\int_0^{l_e} (\varepsilon(\xi))^2 d\xi$$

Axial strain:

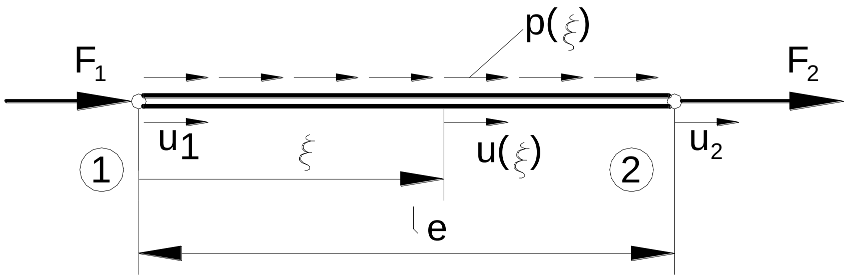
$$\varepsilon(\xi) = \frac{du}{d\xi} = \left\lfloor N_1', N_2' \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e .$$

$$U_e = \frac{EA}{2}\int_0^{l_e} \lfloor q_1, q_2 \rfloor_e \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e d\xi =$$

$$= \frac{EA}{2}\lfloor q_1, q_2 \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1'N_1' & N_1'N_2' \\ N_2'N_1' & N_2'N_2' \end{bmatrix} d\xi \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e ,$$

$$[k]_e = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Equivalent load for an axially loaded rod



uniformly distributed load: $p(\xi)\left[\frac{\mathrm{N}}{\mathrm{m}}\right]$

Approximation of the axial displacements:

$$u(\xi) = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

Work of axial load:

$$W_{ze}^{p} = \int_0^{l_e} p(\xi)u(\xi)d\xi = \int_0^{l_e} \lfloor N_1(\xi)p(\xi), N_2(\xi)p(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e d\xi =$$

$$= \left\lfloor \int_0^{l_e} N_1(\xi)p(\xi), \int_0^{l_e} N_2(\xi)p(\xi)d\xi \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e .$$

$$W_{ze}^{p} = \lfloor F_1^e, F_2^e \rfloor_e \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$$

Equivalent load: $$F_i^e = \int_0^{l_e} N_i(\xi)p(\xi)d\xi$$

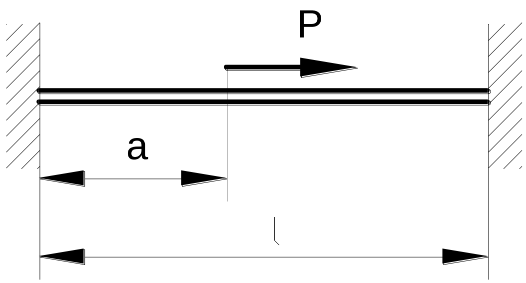
System of equations:

$$[K]\{q\} = \{F\}$$

Stresses in the rod:

$$\sigma = E\varepsilon = E\left\lfloor N_1'(\xi), N_2'(\xi) \right\rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \frac{E(q_2 - q_1)}{l_e}$$

Example 1. A rod loaded with a concentrated force

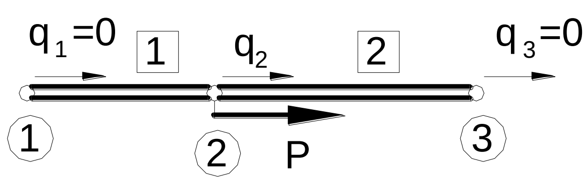


$$\{q\}_1 = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \qquad \{q\}_2 = \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}$$

Local vectors of nodal parameters

Element stiffness matrices:

$$[k]_e^1 = \frac{EA}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^2 = \frac{EA}{l-a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



Global vector of nodal parameters:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_2 \\ 0 \end{Bmatrix}$$

Global load vector:

$$\{F\} = \begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ P \\ R_3 \end{Bmatrix}$$

Global Stiffness Matrix:

$$EA\begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{a}+\frac{1}{l-a} & -\frac{1}{l-a} \\ & -\frac{1}{l-a} & \frac{1}{l-a} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

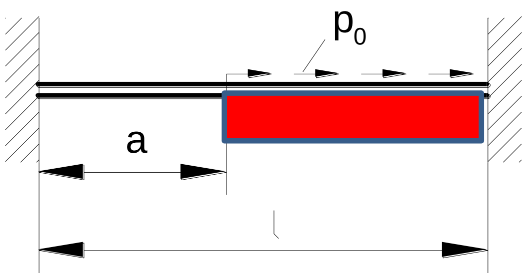
$$F_1 = \frac{-P(l-a)}{l},$$

$$q_2 = \frac{P(l-a)a}{EAl},$$

Reactions

$$F_3 = \frac{-Pa}{l}.$$

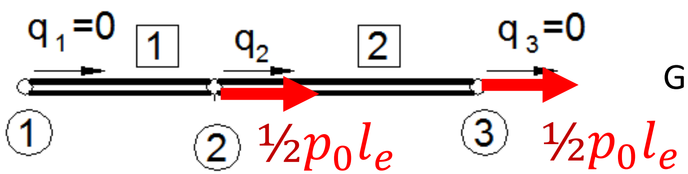
Example 2. A rod loaded with a uniformly distributed load



Global vector of nodal parameters : $\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_2 \\ 0 \end{Bmatrix}$

Equivalent force at node 1 and 2 of element 2:

$$F_1^e = \int_0^{l_e} N_1 p_0 \, d\xi = \int_0^{l_e} \left(1 - \frac{\xi}{l_e}\right) p_0 \, d\xi = \frac{p_0 l_e}{2} = \frac{p_0(l-a)}{2}$$



Global force vector: $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ ½\, p_0(l-a) \\ R_3 + ½\, p_0(l-a) \end{Bmatrix}$

Global Stiffness Matrix:

$$EA \begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{a} + \frac{1}{l-a} & -\frac{1}{l-a} \\ & -\frac{1}{l-a} & \frac{1}{l-a} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$q_2 = \frac{p_0(l-a)^2 a}{2lEA}$$

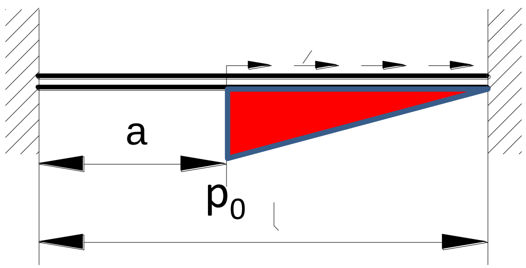
Reaction at node 1

$$F_1 = \frac{-p_0(l-a)^2}{2l}$$

$$F_3 = \frac{-p_0 a(l-a)}{2l}$$

Reaction at node 3: $R_3 = F_3 - \frac{p_0(l-a)l}{2l} = \frac{-p_0 a(l-a)}{2l} - \frac{p_0(l-a)l}{2l} = \frac{-p_0(l-a)(l+a)}{2l}$

Example 3. A rod loaded with a distributed load of a variable linear value



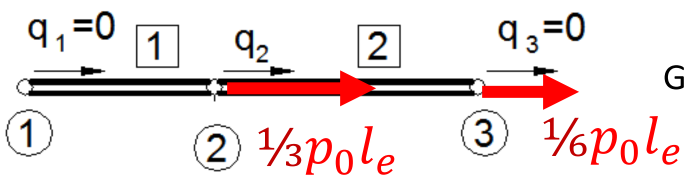
Global vector of nodal parameters : $\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_2 \\ 0 \end{Bmatrix}$

Equivalent force at node 1 of element 2:

$$F_1^e = \int_0^{l_e} N_1 p(\xi)\, d\xi = \int_0^{l_e} N_1 N_1\ p_0\, d\xi = \frac{p_0 l_e}{3} = \frac{p_0(l-a)}{3}$$

Equivalent force at node 2 of element 2:

$$F_2^e = \int_0^{l_e} N_2 p(\xi)\, d\xi = \int_0^{l_e} N_2 N_1\ p_0\, d\xi = \frac{p_0 l_e}{6} = \frac{p_0(l-a)}{6}$$



Global force vector: $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{R_1} \\ \tfrac{1}{3}\, p_0(l-a) \\ \boldsymbol{R_3} + \tfrac{1}{6}\, p_0(l-a) \end{Bmatrix}$

Global Stiffness Matrix:

$$EA\begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{a}+\frac{1}{l-a} & -\frac{1}{l-a} \\ & -\frac{1}{l-a} & \frac{1}{l-a} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$q_2 = \frac{p_0(l-a)^2 a}{3lEA}$$

Reaction at node 1

$$F_1 = \frac{-p_0(l-a)^2}{3l}$$

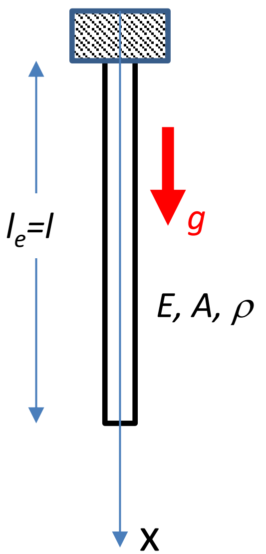
$$F_3 = \frac{-p_0 a(l-a)}{3l}$$

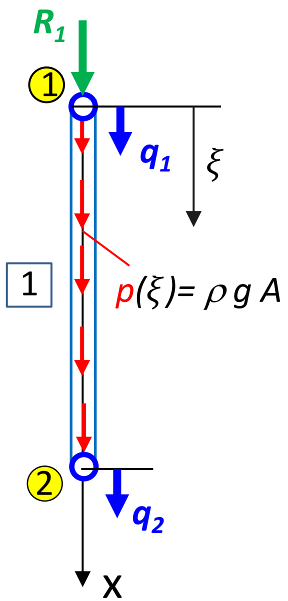
Reaction at node 3:

$$R_3 = F_3 - \frac{p_0(l-a)}{6} = \frac{-p_0 a(l-a)}{3l} - \frac{p_0(l-a)l}{6l} = -\frac{p_0(l-a)(l+2a)}{6l}$$

Example 4. Find the displacements, strain, and stress in a model of a rod hanging in a gravitational field. Use one element, then two. Compare the results with the exact solution.

1) One element:

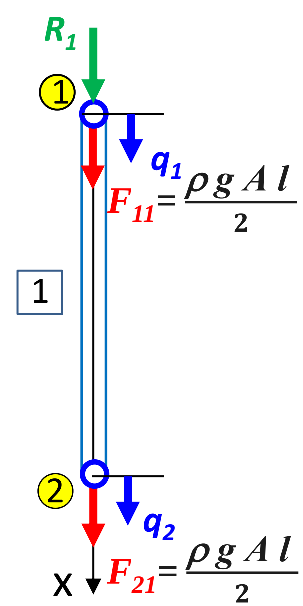




$$\underset{2\times1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1 \qquad \underset{2\times1}{\{q\}} = \{q\}_1$$

$$\underset{2\times2}{[k]_1} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underset{2\times2}{[K]} = \underset{2\times2}{[k]^*_1}$$

$$\underset{1\times2}{\lfloor F \rfloor_1} = \lfloor F_{11},\ F_{21} \rfloor$$



$$F_{11} = \int_0^L p\,N_1(\xi)\,d\xi = \rho g A \int_0^L \left(1-\frac{\xi}{L}\right)d\xi = \rho g A\left(\xi - \frac{\xi^2}{2L}\right)\Big|_0^L = \frac{\rho g A L}{2}$$

$$F_{21} = \int_0^L p\,N_2(\xi)\,d\xi = \frac{\rho g A L}{2} \qquad \underset{1\times2}{\lfloor F \rfloor^e} = \underset{1\times2}{\lfloor F \rfloor_1} = \underset{1\times2}{\lfloor F \rfloor^*_1}$$

$$\lfloor F \rfloor^n_{1\times2} = \lfloor R_1, 0 \rfloor, \quad \lfloor F \rfloor_{1\times2} = \lfloor F \rfloor^e_{1\times2} + \lfloor F \rfloor^n_{1\times2} = \left\lfloor \frac{\rho g A L}{2} + R_1, \frac{\rho g A L}{2} \right\rfloor$$

$$[K]_{2\times2} \cdot \{q\}_{2\times1} = \{F\}_{2\times1} \Rightarrow \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\rho g A L}{2} + R_1 \\ \frac{\rho g A L}{2} \end{Bmatrix}$$

$$\frac{EA}{L} \cdot q_2 = \frac{\rho g A L}{2} \Rightarrow q_2 = \frac{\rho g L^2}{2E}$$

+ boundary condition $\mathbf{q_1=0}$

Displacements in the element:

$$u(\xi) = u(x) = \lfloor N \rfloor \cdot \{q\}_1 = N_1 \cdot q_1 + N_2 \cdot q_2 = \frac{\xi}{L}\frac{\rho g L^2}{2E} = \frac{\rho g L}{2E} \cdot \xi$$

Strain in the element:

$$\varepsilon_x = \frac{q_2 - q_1}{L} = \frac{\rho g L}{2E} = \text{const}$$

Stresses in the element:

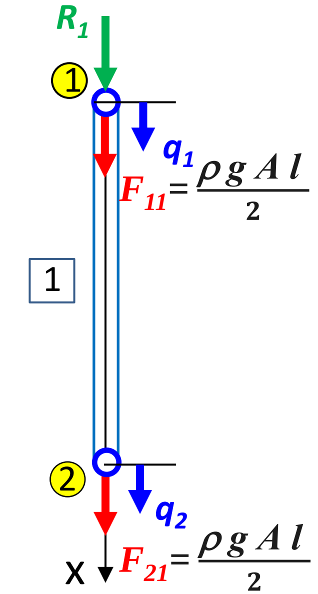
$$\sigma_x = E\varepsilon_x = \frac{\rho g L}{2} = \text{const.}$$

Reactions:

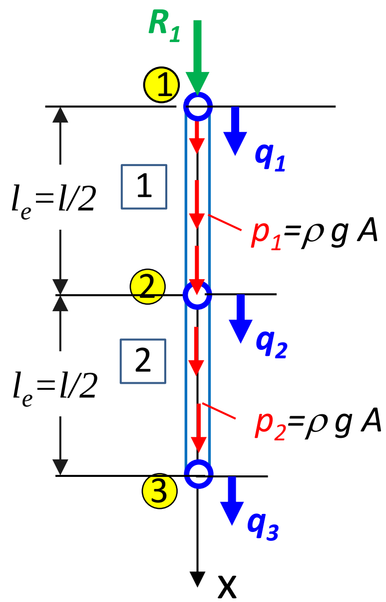
$$\frac{EA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cdot\begin{Bmatrix}0\\ \frac{\rho g l^2}{2E}\end{Bmatrix}=\begin{Bmatrix}\frac{\rho g A l}{2}+R_1\\ \frac{\rho g A l}{2}\end{Bmatrix}$$

$$\frac{EA}{l}\left(1\cdot 0-1\cdot\frac{\rho g l^2}{2E}\right)=\frac{\rho g A l}{2}+R_1$$

$$R_1=-\frac{\rho g A l}{2}-\frac{\rho g A l}{2}=-\rho g A l=-m\cdot g$$



2) Two elements:



Element 1:

$$\underset{2\times1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_1$$

$$\underset{2\times2}{[k]_1} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underset{3\times3}{[k]_1^*} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Element 2:

$$\underset{2\times1}{\{q\}_2} = \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_2$$

$$\underset{2\times2}{[k]_2} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underset{3\times3}{[k]_2^*} = \frac{2EA}{l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underset{3\times1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Global vector
of nodal parameters

Global Stiffness Matrix

$$\underset{3\times3}{[K]} = \underset{3\times3}{[k]_1^*} + \underset{3\times3}{[k]_2^*} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\det \underset{3\times3}{[K]} = 0$$

$$\underset{1\times 2}{\lfloor F \rfloor_1} = \left\lfloor \int_0^{\frac{l}{2}} p_1 N_1 \, d\xi , \int_0^{\frac{l}{2}} p_1 N_2 \, d\xi \right\rfloor = \frac{\rho g A L}{4} [1, 1]$$

$$\underset{1\times 2}{\lfloor F \rfloor_2} = \left\lfloor \int_0^{\frac{l}{2}} p_2 N_1 \, d\xi , \int_0^{\frac{l}{2}} p_2 N_2 \, d\xi \right\rfloor = \frac{\rho g A L}{4} \lfloor 1, 1 \rfloor$$

$$\underset{1\times 3}{\lfloor F \rfloor_1^*} = \frac{\rho g A L}{4} \lfloor 1, 1, 0 \rfloor$$

$$\underset{1\times 3}{\lfloor F \rfloor_2^*} = \frac{\rho g A L}{4} \lfloor 0, 1, 1 \rfloor$$

$$\underset{1\times 3}{\lfloor F \rfloor^e} = \sum_{e=1}^{2} \underset{1\times 3}{\lfloor F \rfloor_e^*} = \underset{1\times 3}{\lfloor F \rfloor_1^*} + \underset{1\times 3}{\lfloor F \rfloor_2^*} = \frac{\rho g A L}{4} \lfloor 1, 2, 1 \rfloor$$

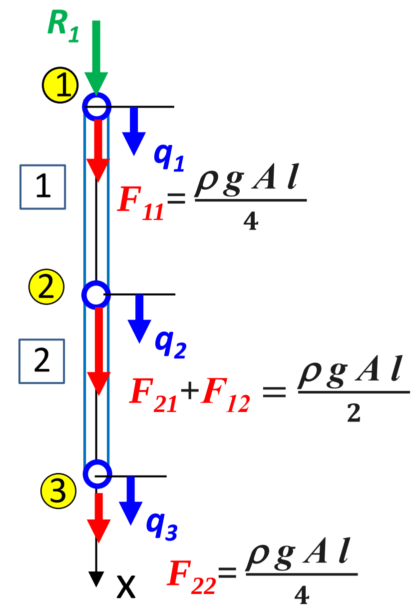
$$\underset{1\times 3}{\lfloor F \rfloor^n} = \lfloor R_1, 0, 0 \rfloor$$

$$\underset{3\times 3}{[K]} \cdot \underset{3\times 1}{\{q\}} = \underset{3\times 1}{\{F\}}$$

$$\underset{1\times 3}{\lfloor F \rfloor} = \left\lfloor R_1 + \frac{\rho g A L}{4}, \frac{\rho g A L}{2}, \frac{\rho g A L}{4} \right\rfloor$$

$$\frac{2EA}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} R_1 + \frac{\rho g A L}{4} \\ \frac{\rho g A L}{2} \\ \frac{\rho g A L}{4} \end{Bmatrix}$$

+ boundary conditions $\mathbf{q_1=0}$



$$\underbrace{\frac{2EA}{l}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}}_{\underset{2\times 2}{[K]}}\cdot\begin{Bmatrix}q_2\\ q_3\end{Bmatrix}=\begin{Bmatrix}\rho g\frac{Al}{2}\\ \rho g\frac{Al}{4}\end{Bmatrix}$$

$$\det\underset{2\times 2}{[K]}=\left(\frac{2EA}{l}\right)^2\cdot\left(2\cdot 1-(-1)\cdot(-1)\right)=\left(\frac{2EA}{l}\right)^2$$

$$\det\underset{2\times 2}{[K]}=\left(\frac{2EA}{l}\right)^2\cdot\left(2\cdot 1-(-1)\cdot(-1)\right)=\left(\frac{2EA}{l}\right)^2$$

$$\underset{2\times 2}{[K^c]}=\frac{2EA}{l}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}=\underset{2\times 2}{[K^c]^T}$$

$$[K]^{-1}=\frac{1}{\det\underset{2\times 2}{[K]}}\cdot\underset{2\times 2}{[K^c]^T}=\frac{l}{2EA}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

$$\begin{Bmatrix}q_2\\ q_3\end{Bmatrix}=[K]^{-1}\cdot\frac{\rho g A l}{4}\begin{Bmatrix}2\\ 1\end{Bmatrix}=\frac{\rho g A l^2}{8EA}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\cdot\begin{Bmatrix}2\\ 1\end{Bmatrix}=\frac{\rho g l^2}{8E}\begin{Bmatrix}3\\ 4\end{Bmatrix}$$

$$q_2=\frac{3}{8}\frac{\rho g l^2}{E}$$

$$q_3=\frac{\rho g l^2}{2E}$$

Displacements in elements (FEM solution in elements):

$$\boxed{1}: \quad u_1(\xi) = \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_1} = \left(1 - \frac{\xi}{\left(\frac{L}{2}\right)}\right) \cdot q_1 + \frac{\xi}{\left(\frac{L}{2}\right)} \cdot q_2 =$$

$$= \frac{2\xi}{L} \cdot \frac{3}{8} \frac{\rho g L^2}{E} = \frac{3 \rho g L}{4E} \cdot \xi$$

$$x \in \langle 0, \tfrac{L}{2} \rangle; \quad \xi = x \longrightarrow u(x) = \frac{3\rho g L}{4E} \cdot x$$

$$u(0) = 0, \quad u\left(\tfrac{L}{2}\right) = \frac{3}{8} \frac{\rho g L^2}{E}$$

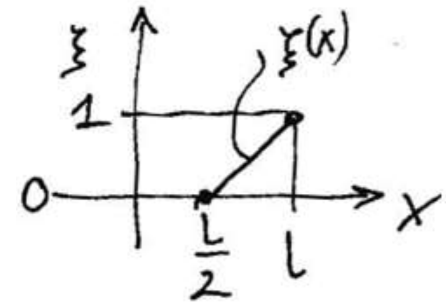
$$\varepsilon_{x_1} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_1} = -\frac{2}{L} \cdot q_1 + \frac{2}{L} \cdot q_2 = \frac{3\rho g L}{4E}$$

$$\sigma_{x_1} = E \varepsilon_{x_1} = \frac{3 \rho g L}{4}$$

$$\boxed{2} \quad u_2(\xi) = \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_2} = \left(1 - \frac{2\xi}{l}\right) q_2 + \frac{2\xi}{l} \cdot q_3 =$$

$$= \frac{3}{8}\frac{\rho g l^2}{E} - \frac{3\rho g l}{4E}\cdot\xi + \frac{\rho g l}{E}\cdot\xi = \frac{\rho g l}{4E}\cdot\xi + \frac{3\rho g l^2}{8E}$$

$$x \in \left\langle \frac{l}{2}, l \right\rangle \; ; \; \xi(x) = x - \frac{l}{2}$$



$$u_2(x) = \frac{\rho g l}{4E}\cdot x + \frac{\rho g l^2}{4E} = \frac{\rho g}{4E}\left(x\cdot l + l^2\right)$$

$$u\left(\frac{l}{2}\right) = \frac{3}{8}\frac{\rho g l^2}{E} \quad , \quad u(l) = \frac{\rho g l^2}{2E}$$

$$\varepsilon_{x_2} = \left\lfloor \frac{dN_1}{d\xi} , \frac{dN_2}{d\xi} \right\rfloor \cdot \{q\}_2 = -\frac{2}{l} q_2 + \frac{2}{l} q_3 = \frac{2(q_3 - q_2)}{l} = \frac{\rho g l}{4E}$$

$$\sigma_{x_2} = E\,\varepsilon_{x_2} = \frac{\rho g l}{4}$$

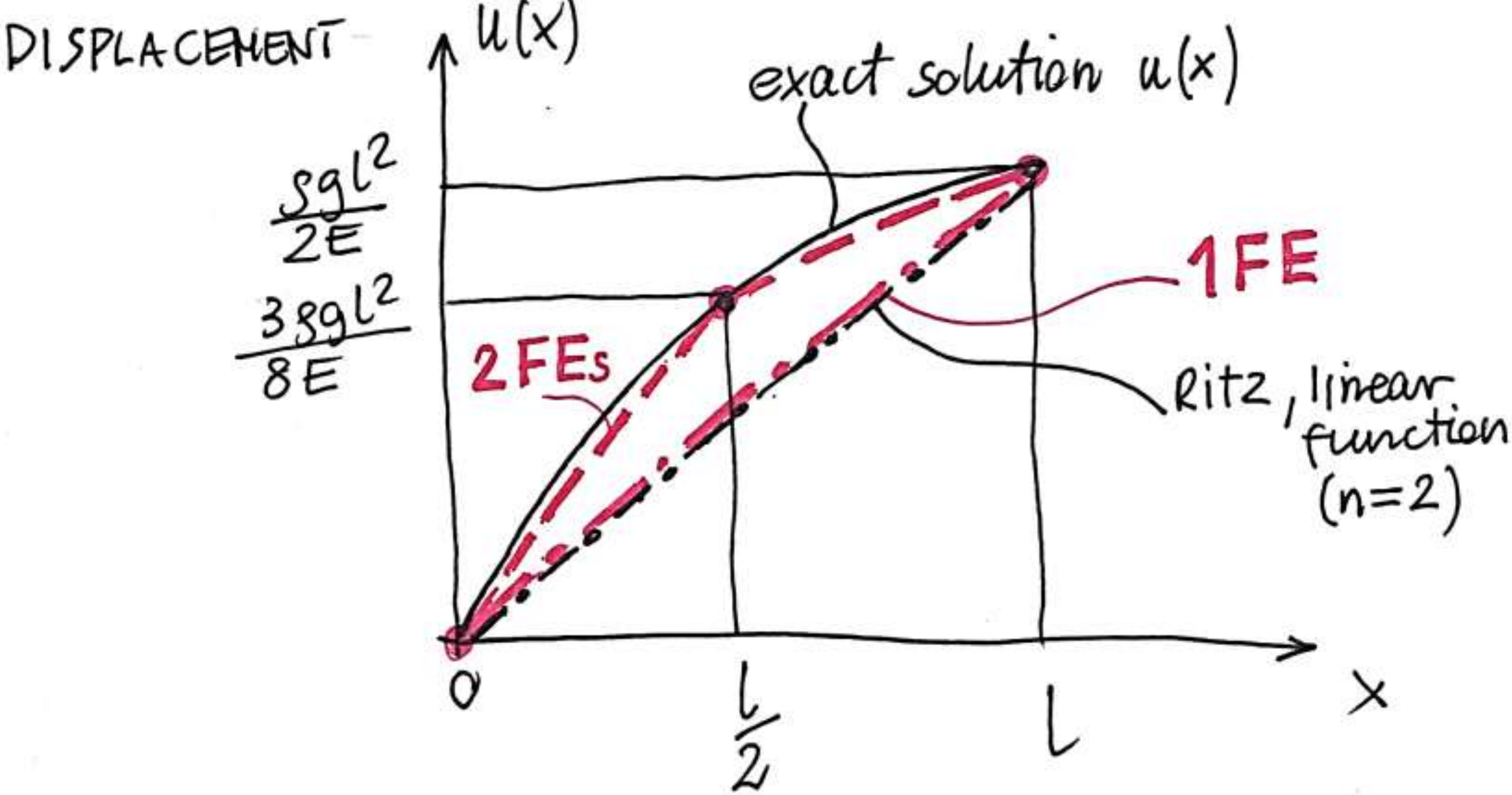
Reactions:

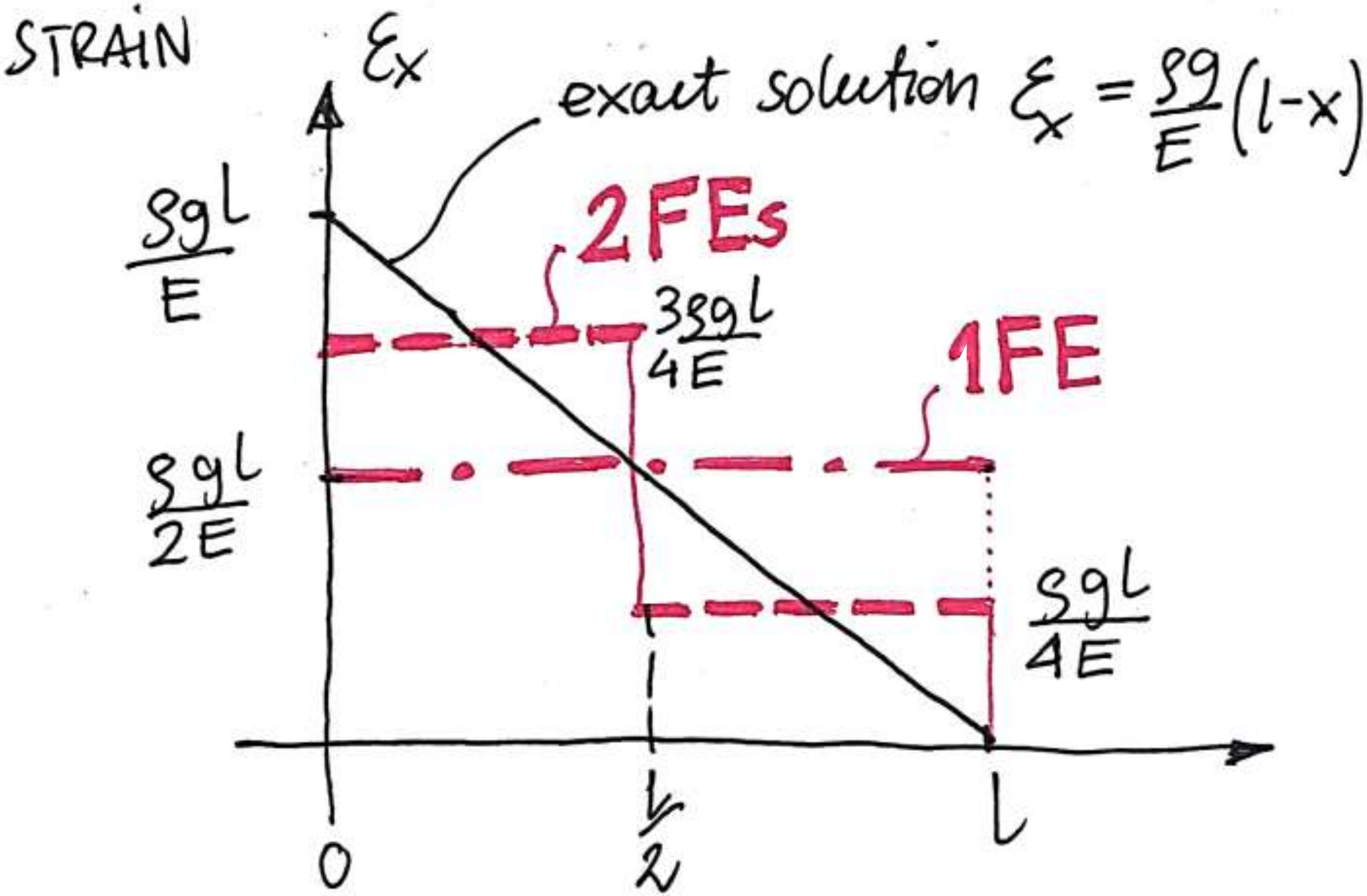
$$\frac{2EA}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\cdot\frac{\rho g l^2}{8E}\begin{Bmatrix}0\\3\\4\end{Bmatrix}=\begin{Bmatrix}R_1+\frac{\rho g A l}{4}\\ \vdots\\ \vdots\end{Bmatrix}$$

$$\frac{2EA}{l}\left(1\cdot 0-1\cdot 3+0\cdot 4\right)\cdot\frac{\rho g l^2}{8E}=R_1+\rho g\frac{Al}{4}$$

$$-\frac{3}{4}\rho g A l-\frac{1}{4}\rho g A l=R_1$$

$$R_1=-\rho g A l=-mg$$


DISPLACEMENT
u(x)
exact solution u(x)
$\frac{\rho g l^2}{2E}$
$\frac{3\rho g l^2}{8E}$
1FE
2FEs
Ritz, linear function (n=2)
0
$\frac{l}{2}$
l
x



STRAIN
ε_x
exact solution $\varepsilon_x = \frac{\rho g}{E}(l-x)$
$\frac{\rho g l}{E}$
2FEs
$\frac{3\rho g l}{4E}$
1FE
$\frac{\rho g l}{2E}$
$\frac{\rho g l}{4E}$
0
$l/2$
l

10 finite elements